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ScienceDirect
Journal of Geometry and Physics 57 (2007) 2114–2123
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Mapped null hypersurfaces and Legendrian maps

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Received 25 February 2007; accepted 22 May 2007
Available online 25 May 2007

Abstract

For an $(m+1)$-dimensional space–time (X^{m+1}, g), define a mapped null hypersurface to be a smooth map $\nu : \mathcal{N}^m \to \mathsf{X}^{m+1}$ (that is not necessarily an immersion) such that there exists a smooth field of null lines along ν that are both tangent and g-orthogonal to ν.

We study relations between mapped null hypersurfaces and Legendrian maps to the spherical cotangent bundle $ST^*\mathcal{M}$ of an immersed spacelike hypersurface $\mu : \mathcal{M}^m \to \mathsf{X}^{m+1}$. We show that a Legendrian map $\widetilde{\lambda} : \mathcal{L}^{m-1} \to (ST^*\mathcal{M})^{2m-1}$ defines a mapped null hypersurface in X. On the other hand, the intersection of a mapped null hypersurface $\nu : \mathcal{N}^m \to \mathsf{X}^{m+1}$ with an immersed spacelike hypersurface $\mu' : \mathcal{M}'^m \to \mathsf{X}^{m+1}$ defines a Legendrian map to the spherical cotangent bundle $ST^*\mathcal{M}'$. This map is a Legendrian immersion if ν came from a Legendrian immersion to $ST^*\mathcal{M}$ for some immersed spacelike hypersurface $\mu : \mathcal{M}^m \to \mathsf{X}^{m+1}$.

MSC: primary 53C50; secondary 57R17

Keywords: Lorentz manifold; Hypersurface; Null surface; Null cone; Contact manifold; Legendrian manifold

We work in the C^∞ category, and the word "smooth" means C^∞. The manifolds in this work are assumed to be smooth without boundary. They are not assumed to be oriented, or connected, or compact unless the opposite is explicitly stated. In this work (X^{m+1}, g) is an $(m+1)$-dimensional Lorentzian manifold that is not assumed to be geodesically complete.

A "vector field" on a manifold Y is a smooth section of the tangent bundle $\tau_Y : TY \to Y$, and a "vector field along a map" $\phi : Y_1 \to Y_2$ of one manifold to another is a smooth map $\Phi : Y_1 \to TY_2$ such that $\phi = \tau_{Y_2} \circ \Phi$. Covector fields and line fields on a manifold and along a map ϕ are defined in a similar way.

1. Preliminaries

Let us recall some basic Lorentz geometry facts. Put Ξ to be the space of vector fields on X. There exists a unique connection ∇^g on X that satisfies the following metric compatibility and torsion free conditions:

$$\begin{aligned} &\xi_1 g(\xi_2, \xi_3) = g(\nabla^g_{\xi_1}\xi_2, \xi_3) + g(\xi_2, \nabla^g_{\xi_1}\xi_3), \\ &[\xi_1, \xi_2] = \nabla^g_{\xi_1}\xi_2 - \nabla^g_{\xi_2}\xi_1, \end{aligned} \tag{1.1}$$

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doi:10.1016/j.geomphys.2007.05.005

for all $\xi_1, \xi_2, \xi_3 \in \Xi$; see [1, p. 22]. This connection is called a *Levi-Civita connection.* When no confusion arises we will write ∇ instead of ∇^g.

A *geodesic* $c : (a, b) \to (\mathsf{X}, g)$ is a smooth curve satisfying $\nabla_{c'} c' = 0$, for all of its points. Similarly to the Riemannian case, one uses geodesics to define the exponential $\exp_x : T_x\mathsf{X} \to \mathsf{X}$. Note that $\exp_x$ is defined on the star-convex, with respect to $\mathbf{0}$, domain of $T_x\mathsf{X}$, rather than on the whole $T_x\mathsf{X}$. There is an open neighborhood $V_x \subset T_x\mathsf{X}$ of $\mathbf{0}$ such that $\exp_x |_{V_x}$ is a diffeomorphism. The open set $U_x = \exp_x(V_x)$ is called a *normal neighborhood* of x. A neighborhood is *geodesically convex* if any two of its points can be joined by a unique geodesic arc inside of it. The result of Whitehead [12,13], [11, Section 5, Proposition 7] is that every point in a semi-Riemannian, and hence Lorentzian manifold has a geodesically convex normal neighborhood. A *simple region* is a geodesically convex normal neighborhood with compact closure whose boundary is diffeomorphic to S^m.

A non-zero vector $v \in T\mathsf{X}$ is called *spacelike, non-spacelike, null (lightlike), or timelike* if $g(v, v)$ is positive, non-positive, zero, or negative, respectively. A piecewise smooth curve is called *spacelike, non-spacelike, null*, or *timelike* if all of its velocity vectors are respectively spacelike, non-spacelike, null, or timelike. For a point x in a Lorentz (X, g) the set of all non-spacelike vectors in $T_x\mathsf{X}$ consists of two connected components that are hemicones. A continuous with respect to $x \in \mathsf{X}$ choice of one of the two hemicones is called the *time orientation* of X. The non-spacelike vectors from these chosen hemicones are called *future pointing*. A time oriented (X^{m+1}, g) is called a *space–time*.

An immersion $\kappa : \mathcal{K}^k \to \mathsf{X}^{m+1}$ of a k-manifold is said to be an immersed spacelike or timelike submanifold if the pull back of g to $T\mathcal{K}$ is respectively a Riemannian or a Lorentzian metric. An immersion (respectively an embedding) $i : \mathcal{H}^m \to \mathsf{X}^{m+1}$ of an m-manifold is called an *immersed (respectively an embedded) hypersurface.* An immersed hypersurface is called an *immersed null hypersurface* if for every $h \in \mathcal{H}$ the pull back of g is degenerate on $T_h\mathcal{H}$. Similarly one defines *embedded null hypersurfaces* and immersed and embedded *spacelike and timelike hypersurfaces*.

An immersed (or an embedded) hypersurface $i : \mathcal{H}^m \to \mathsf{X}^{m+1}$ can be canonically equipped with a line field $L_h \subset T_{i(h)}\mathsf{X}$, $h \in \mathcal{H}$, along i such that for every $h \in \mathcal{H}$ the line L_h is g-orthogonal to $i_*(T_h\mathcal{H}) \subset T_{i(h)}\mathsf{X}$. It is easy to verify that an immersed hypersurface is spacelike, timelike or null if and only if for every $h \in \mathcal{H}$ the non-zero vectors in L_h are respectively timelike, spacelike or null. Since the Lorentz metric is non-degenerate, for an immersed null hypersurface the line field L_h is tangent to $i(\mathcal{H})$, i.e. $L_h \subset i_*(T_h\mathcal{H})$ for all $h \in \mathcal{H}$. This observation motivates the following definition.

Definition 1.1 (*Mapped Null Hypersurface*). A smooth map $\nu : \mathcal{N}^m \to \mathsf{X}^{m+1}$ of an m-manifold is called *a mapped null hypersurface* if there exists a smooth (non-oriented) line field $L_n \subset T_{\nu(n)}\mathsf{X}$, $n \in \mathcal{N}$, along ν such that for every $n \in \mathcal{N}$ the non-zero vectors in L_n are null, $L_n \subset \nu_*(T_n\mathcal{N})$, and L_n is g-orthogonal to $\nu_*(T_n\mathcal{N}) \subset T_{\nu(n)}\mathsf{X}$.

Two null vectors are orthogonal if and only if one of them is a multiple of the other. Hence the line field L_n in the above definition is completely determined by the map ν.

Every immersed null hypersurface is a mapped null hypersurface. However mapped null hypersurfaces can be quite singular. For example if Y^{m-1} is an $(m-1)$-manifold and $\gamma : \mathbb{R} \to \mathsf{X}$ is a curve such that $\gamma'(t)$ is null and non-zero for all t, then the composition of γ and of the projection $Y^{m-1} \times \mathbb{R} \to \mathbb{R}$ gives a mapped null hypersurface $Y^{m-1} \times \mathbb{R} \to \mathsf{X}^{m+1}$.

Let us recall some basic contact geometry facts. Let Q^{2k-1} be a smooth $(2k-1)$-dimensional manifold equipped with a smooth (non-oriented) hyperplane field $\zeta = \{\zeta_q^{2k-2} \subset T_q Q^{2k-1} \mid q \in Q\}$. This hyperplane field is called *a contact structure*, if it can be locally presented as the kernel of a 1-form α with nowhere zero $\alpha \wedge (d\alpha)^{k-1}$.

An immersion (respectively an embedding) $i : \mathcal{L}^{k-1} \to Q^{2k-1}$ of a $(k-1)$-dimensional manifold $\mathcal{L}^{k-1}$ into a contact manifold (Q^{2k-1}, ζ) is called a *Legendrian immersion (respectively a Legendrian embedding)*, if $i_*(T_l\mathcal{L}) \subset \zeta_{i(l)}$, for all $l \in \mathcal{L}$.

Definition 1.2 (*Legendrian Map*). We say that a smooth map $\widetilde{\lambda} : \mathcal{L}^{k-1} \to Q^{2k-1}$ to a contact (Q^{2k-1}, ζ) is a *Legendrian map* if $\widetilde{\lambda}_*(T_l\mathcal{L}) \subset \zeta_{\widetilde{\lambda}(l)}$, for all $l \in \mathcal{L}$. Every Legendrian immersion is a Legendrian map. However a Legendrian map can be quite singular and the trivial map $\mathcal{L}^{k-1} \to \mathrm{pt} \in Q^{2k-1}$ is a Legendrian map.

Example 1.3 (*The Natural Contact Structure on $ST^*\mathcal{M}$*). For a smooth manifold $\mathcal{M}^k$, put $ST^*\mathcal{M}$ to be the spherical cotangent bundle, i.e. the quotient of $T^*\mathcal{M}$ minus the zero section by the action of the group $\mathbb{R}^+$ of positive real numbers under multiplication. Put $\mathrm{pr} = \mathrm{pr}_M : ST^*\mathcal{M} \to \mathcal{M}$ to be the corresponding S^{k-1}-bundle map. A point $p \in ST^*\mathcal{M}$ is the equivalence class of non-zero linear functionals on $T_{\mathrm{pr}\,p}\mathcal{M}$. Two functionals are equivalent if and only if their kernels are equal and the half-spaces of $T_{\mathrm{pr}\,p}\mathcal{M}$ where the functionals are positive are equal. Thus p is completely determined by the hyperplane $\ker p \subset T_{\mathrm{pr}(p)}\mathcal{M}$ together with the half-space of $T_{\mathrm{pr}\,p}\mathcal{M} \setminus \ker p$ where the functionals are positive.

The natural contact structure

$$\zeta = \{\zeta_p^{2k-2} \subset T_p(ST^*\mathcal{M})^{2k-1}, p \in ST^*\mathcal{M}\}$$

is given by $\zeta_p = (\mathrm{pr}_*)^{-1}(\ker p)$.

A map $\widetilde{\lambda} : \mathcal{L}^{k-1} \to (ST^*\mathcal{M})^{2k-1}$ can be described as the pair consisting of the smooth map $\lambda = \mathrm{pr} \circ \widetilde{\lambda}$ and a smooth nowhere zero covector field $\theta_l \in T^*_{\lambda(l)}\mathcal{M}, l \in \mathcal{L}$, along λ such that for every $l \in \mathcal{L}$ the equivalence class of θ_l is $\widetilde{\lambda}(l)$. The covector field θ_l is defined uniquely up to a multiplication by a positive smooth function $\mathcal{L} \to \mathbb{R}$.

Clearly $\widetilde{\lambda}$ is a Legendrian map if and only if $\lambda_*(T_l\mathcal{L}) \subset \ker \theta_l$, for all $l \in \mathcal{L}$.

If $\mathcal{M}$ is equipped with a Riemannian or Lorentzian metric h, then we can identify the tangent and the cotangent bundles of $\mathcal{M}$ and we can identify the spherical tangent and the spherical cotangent bundles $\mathrm{pr} : ST\mathcal{M} \to \mathcal{M}$ and $\mathrm{pr} : ST^*\mathcal{M} \to \mathcal{M}$. Thus a smooth map $\widetilde{\lambda} : \mathcal{L} \to ST\mathcal{M} = ST^*\mathcal{M}$ can be described as the pair consisting of the smooth map $\lambda = \mathrm{pr} \circ \widetilde{\lambda}$ and a smooth nowhere zero vector field $X_l \subset T_{\lambda(l)}\mathcal{M}, l \in \mathcal{L}$, along λ such that for every $l \in \mathcal{L}$ the equivalence class of X_l is $\widetilde{\lambda}(l)$. Clearly $\widetilde{\lambda}$ is a Legendrian map if and only if $X(l)$ is h-orthogonal to $\lambda_*(T_l\mathcal{L})$, for all $l \in \mathcal{L}$.

Now let (X^{m+1}, g) be a space–time and let $\nu : \mathcal{N}^m \to \mathsf{X}^{m+1}$ be a mapped null hypersurface. Let $L_n \in T_{\nu(n)}\mathsf{X}, n \in \mathcal{N}$, be the unique smooth non-oriented line field along ν from the definition of the mapped null hypersurface. Since (X, g) is time oriented, we can orient the null lines L_n in the direction of the future. This oriented line field defines the Legendrian map $\widetilde{\nu} : \mathcal{N} \to ST\mathsf{X}$ such that $\mathrm{pr}_{\mathsf{X}} \circ \widetilde{\nu} = \nu$.

2. From Legendrian maps to mapped null hypersurfaces

Let (X^{m+1}, g) be a space–time, let $\mu : \mathcal{M}^m \to \mathsf{X}^{m+1}$ be an immersed spacelike hypersurface, and let $\overline{g}$ be the induced Riemannian metric on $\mathcal{M}$. Let $\widetilde{\lambda} : \mathcal{L}^{m-1} \to ST^*M = STM$ be a Legendrian map that is described by the pair $\lambda = \mathrm{pr} \circ \lambda$ and the smooth unit length vector field $X_l \in T_{\lambda(l)}\mathcal{M}, l \in \mathcal{L}$, along λ.

Since the immersed hypersurface μ is spacelike, for each $l \in \mathcal{L}$ the space $T_{\mu\circ\lambda(l)}\mathsf{X}$ splits as the direct sum of $\mu_*(T_{\lambda(l)}\mathcal{M})$ and its one-dimensional g-orthogonal compliment $(\mu_*(T_{\lambda(l)}\mathcal{M}))^\perp$ consisting of timelike vectors. Thus for each $l \in \mathcal{L}$, there exists the unique future pointing null vector $N_l = (N_l^s, N_l^t) \in \mu_*(T_{\lambda(l)}\mathcal{M}) \oplus (\mu_*(T_{\lambda(l)}\mathcal{M}))^\perp = T_{\mu\circ\lambda(l)}\mathsf{X}$ such that $N_l^s = \mu_*(X_l)$. Put $\gamma_l(t)$ to be the maximal null geodesic such that $\gamma_l'(0) = N_l$.

We get the map from a subset of $\mathcal{L} \times \mathbb{R}$ to X defined as $(l, t) \to \gamma_l(t)$, for $l \in \mathcal{L}$ and t in the domain of the null geodesic γ_l. Since each point of X has a geodesically convex normal neighborhood, the above map is defined on an open neighborhood of $\mathcal{L} \times 0 \subset \mathcal{L} \times \mathbb{R}$. Put $\mathcal{N} \subset \mathcal{L}^{m-1} \times \mathbb{R}$ to be the maximal open neighborhood where the map is defined and put $\nu : \mathcal{N}^m \to \mathsf{X}^{m+1}$ to be the resulting map.

It is easy to see that $\gamma_l(t)$ is a future directed null geodesic such that $\gamma_l'(0)$ is g-orthogonal to $(\mu \circ \lambda)_*(T_l\mathcal{L}) \subset T_{\mu\circ\lambda(l)}\mathsf{X}$. Thus ν is a mapped hypersurface corresponding to a congruence of such null geodesics.

If $\lambda : \mathcal{L}^{m-1} \to \mathcal{M}^m$ is an immersion whose normal bundle is orientable, then there are exactly two unit lengths vector fields that are $\overline{g}$-orthogonal to λ and they define two Legendrian immersions $\mathcal{L} \to ST\mathcal{M}$. The union of the mapped hypersurfaces ν constructed for these two Legendrian immersions should be thought of as the wave front associated to $\mu \circ \lambda(\mathcal{L})$.

Theorem 2.1. *Let (X^{m+1}, g) be a space–time, let $\mu : \mathcal{M}^m \to \mathsf{X}$ be an immersed spacelike hypersurface, and let $\widetilde{\lambda} : \mathcal{L}^{m-1} \to ST^*\mathcal{M}$ be a Legendrian map. Let $\nu : \mathcal{N}^m \to \mathsf{X}^{m+1}$ be the map obtained as above from μ and $\widetilde{\lambda}$. Then the following two statements hold:*

1. *ν is a mapped null hypersurface. In particular, the map $\widetilde{\nu} : \mathcal{N} \to ST\mathsf{X} = ST^*\mathsf{X}$ that sends $(l, t) \in \mathcal{N} \subset \mathcal{L} \times \mathbb{R}$ to the direction of $\gamma_l'(t)$ is a Legendrian map such that $\mathrm{pr}_{\mathsf{X}} \circ \widetilde{\nu} = \nu$, see Example 1.3.*
2. *If $\widetilde{\lambda}$ is a Legendrian immersion, then $\widetilde{\nu}$ also is a Legendrian immersion.*

Proof. *Let us prove statement* 1 *of the theorem.* We have that $\mathcal{L} \times \{0\} \subset \mathcal{N} \subset \mathcal{L} \times \mathbb{R}$ and that $\mathcal{N} \cap (l \times \mathbb{R})$ is connected, for all $l \in \mathcal{L}$. The map ν is smooth, since the velocity vectors $\gamma_l'(0) = N_l$ smoothly depend on $l \in \mathcal{L}$ and since $\nu(l, t) = \gamma_l(t)$.

Consider the vector field $\widetilde{N} = \widetilde{N}_n = (\mathbf{0}, \frac{\partial}{\partial t})$ on $\mathcal{N} \subset \mathcal{L} \times \mathbb{R}$. Define the vector field $N_n, n = (l, t) \in \mathcal{N} \subset \mathcal{L} \times \mathbb{R}$, along ν via $N_n = \nu_*(\widetilde{N}_n) = \nu_*(l, t)(\mathbf{0}, \frac{\partial}{\partial t}) \in T_{\nu(n)}\mathsf{X}$. Clearly $N_n = \gamma_l'(t)$. Also $N_{(l,0)} = N_l$ for all $l \in \mathcal{L}$.

Put $L_n \subset T_{\nu(n)}\mathsf{X}$ to be the line generated by $\gamma_l'(t)$. We get the smooth line field $L_n, n \in \mathcal{N}$, along ν. Since γ_l are null geodesics, all the non-zero vectors in the lines L_n are null. Also $L_n \subset \nu_*(T_n\mathcal{N})$ by construction.

Thus to prove the theorem it suffices to show that $g\left(N_n, \nu_*(\widetilde{Z}_n)\right) = 0$ for all $n \in \mathcal{N}$, $\widetilde{Z}_n \in T_n\mathcal{N}$. Fix $n_0 = (l_0, t_0)$, and $\widetilde{Z}_{n_0} \in T_{n_0}\mathcal{N}$. Extend $\widetilde{Z}_{n_0}$ to a smooth vector field $\widetilde{Z} = \widetilde{Z}_n, n \in \mathcal{N}$, on $\mathcal{N}$ such that $[\widetilde{N}, \widetilde{Z}]$ vanishes in a neighborhood of $(l_0 \times \mathbb{R}) \cap \mathcal{N}$.

Consider the following commutative diagram:

$$\begin{array}{ccccc}
(T\mathcal{N}, \widetilde{g}) & \xrightarrow{j} & (\nu^* T\mathsf{X}, \widehat{g}, \nabla^{\widehat{g}}) & \xrightarrow{F} & (T\mathsf{X}, g, \nabla^g) \\
\Big\downarrow{\scriptstyle \tau_{\mathcal{N}}} & & \Big\downarrow{\scriptstyle \nu^*\tau_{\mathsf{X}}} & & \Big\downarrow{\scriptstyle \tau_{\mathsf{X}}} \\
\mathcal{N} & \xrightarrow{\text{id}} & \mathcal{N} & \xrightarrow{\nu} & \mathsf{X}.
\end{array} \tag{2.1}$$

Here $\tau_{\mathsf{X}} : T\mathsf{X} \to \mathsf{X}$ is the tangent bundle, $\nu^*\tau_{\mathsf{X}} : \nu^*T\mathsf{X} \to \mathcal{N}$ is the induced bundle, $\widehat{g} = \nu^* g$ is the induced tensor field on $\nu^*T\mathsf{X}$, $\nabla^{\widehat{g}}$ is the connection on $\nu^*\tau_{\mathsf{X}}$ induced from ∇^g, $j : T\mathcal{N} \to \nu^*T\mathsf{X}$ is the natural bundle map, and $\widetilde{g} = \nu^* g$ is the induced tensor field on $T\mathcal{N}$. Put $\widehat{N} = j(\widetilde{N})$ and $\widehat{Z} = j(\widetilde{Z})$ to be the sections of the vector bundle $\nu^*\tau_{\mathsf{X}}$.

Let T be the torsion tensor field of ∇^g and let $\widetilde{W}_i : \mathcal{N} \to T\mathcal{N}, i = 1, 2$, be vector fields. We have $T(\nu_*(\widetilde{W}_1), \nu_*(\widetilde{W}_2)) = \nabla^{\widehat{g}}_{\widetilde{W}_1} j(\widetilde{W}_2) - \nabla^{\widehat{g}}_{\widetilde{W}_2} j(\widetilde{W}_1) - j([\widetilde{W}_1, \widetilde{W}_2])$, see [4, Lemma in Section 2.5]. Since ∇^g is torsion free, we have

$$\nabla^{\widehat{g}}_{\widetilde{W}_1} j(\widetilde{W}_2) - \nabla^{\widehat{g}}_{\widetilde{W}_2} j(\widetilde{W}_1) = j([\widetilde{W}_1, \widetilde{W}_2]), \tag{2.2}$$

for every two smooth vector fields $\widetilde{W}_1, \widetilde{W}_2 : \mathcal{N} \to T\mathcal{N}$.

Since ∇^g is compatible with g, we have

$$\widetilde{W}\widehat{g}(\widehat{Z}_1, \widehat{Z}_2) = \widehat{g}\left(\nabla^{\widehat{g}}_{\widetilde{W}}\widehat{Z}_1, \widehat{Z}_2\right) + \widehat{g}\left(\widehat{Z}_1, \nabla^{\widehat{g}}_{\widetilde{W}}\widehat{Z}_2\right), \tag{2.3}$$

for every vector field $\widetilde{W} : \mathcal{N} \to T\mathcal{N}$ and every two sections $\widehat{Z}_1, \widehat{Z}_2$ of $\nu^*\tau_{\mathsf{X}} : \nu^*T\mathsf{X} \to \mathcal{N}$. This identity (2.3) is proved in [4, Lemma in Section 3.4] for connections induced from connections compatible with a Riemannian metric. However the same proof works for connections compatible with a Lorentzian metric.

Clearly, $g\left(N_{n_0}, \nu_*(\widetilde{Z}_{n_0})\right) = \widetilde{g}\left(\widetilde{N}_{n_0}, \widetilde{Z}_{n_0}\right)$. Using identity (2.3) and the fact that the vectors $N_{(l,t)}$ are the velocity vectors $\gamma_l'(t)$ of the null geodesics, we have

$$\widetilde{N}\widetilde{g}(\widetilde{N}, \widetilde{Z}) = \widetilde{N}\widehat{g}(\widehat{N}, \widehat{Z}) = \widehat{g}(\nabla^{\widehat{g}}_{\widetilde{N}}\widehat{N}, \widehat{Z}) + \widehat{g}(\widehat{N}, \nabla^{\widehat{g}}_{\widetilde{N}}\widehat{Z}) = 0 + \widehat{g}(\widehat{N}, \nabla^{\widehat{g}}_{\widetilde{N}}\widehat{Z}). \tag{2.4}$$

Using identities (2.2) and (2.3) and the fact that the vectors N_n are null, we have

$$\begin{aligned}
\widehat{g}(\widehat{N}, \nabla^{\widehat{g}}_{\widetilde{N}}\widehat{Z}) &= \widehat{g}\left(\widehat{N}, \nabla^{\widehat{g}}_{\widetilde{Z}}\widehat{N} + j([\widetilde{N}, \widetilde{Z}])\right) = \widehat{g}\left(\widehat{N}, \nabla^{\widehat{g}}_{\widetilde{Z}}\widehat{N} + 0\right) = \frac{1}{2}\widehat{g}(\nabla^{\widehat{g}}_{\widetilde{Z}}\widehat{N}, \widehat{N}) + \frac{1}{2}\widehat{g}(\widehat{N}, \nabla^{\widehat{g}}_{\widetilde{Z}}\widehat{N}) \\
&= \frac{1}{2}\widetilde{Z}\widehat{g}(\widehat{N}, \widehat{N}) = \frac{1}{2}\widetilde{Z}0 = 0.
\end{aligned} \tag{2.5}$$

Combining Eqs. (2.4) and (2.5) we have $\widetilde{N}\widetilde{g}(\widetilde{N}, \widetilde{Z}) = 0$. Since $\widetilde{N} = (\mathbf{0}, \frac{\partial}{\partial t})$, we have

$$\widetilde{g}(\widetilde{N}_{n_0}, \widetilde{Z}_{n_0}) = \widetilde{g}(\widetilde{N}_{(l_0,t_0)}, \widetilde{Z}_{(l_0,t_0)}) = \widetilde{g}(\widetilde{N}_{(l_0,0)}, \widetilde{Z}_{(l_0,0)}). \tag{2.6}$$

Decompose $\widetilde{Z}_{(l_0,0)}$ as $\widetilde{Z}^{\mathcal{L}}_{(l_0,0)} + r\widetilde{N}_{(l_0,0)}$, with $\widetilde{Z}^{\mathcal{L}}_{(l_0,0)} \in T_{(l_0,0)}(L \times 0)$. We identify $T_{(l_0,0)}(\mathcal{L} \times 0)$ with $T_{l_0}\mathcal{L}$ and we denote by $\widetilde{Z}^{\mathcal{L}}_{l_0} \in T_{l_0}\mathcal{L}$ the vector corresponding to $\widetilde{Z}^{\mathcal{L}}_{(l_0,0)} \in T_{(l_0,0)}(\mathcal{L} \times 0)$. We have

$$\begin{aligned}\widetilde{g}(\widetilde{N}_{(l_0,0)}, \widetilde{Z}_{(l_0,0)}) &= \widetilde{g}\left(\widetilde{N}_{(l_0,0)}, \widetilde{Z}^{\mathcal{L}}_{(l_0,0)} + r\widetilde{N}_{(l_0,0)}\right)\\ &= rg\left(\nu_*(\widetilde{N}_{(l_0,0)}), \nu_*(\widetilde{N}_{(l_0,0)})\right) + g\left(\nu_*(\widetilde{N}_{(l_0,0)}), \nu_*(\widetilde{Z}^{\mathcal{L}}_{(l_0,0)})\right)\\ &= rg(N_{l_0}, N_{l_0}) + g\left(N_{l_0}, (\mu\circ\lambda)_*(\widetilde{Z}^{\mathcal{L}}_{l_0})\right) = 0 + g\left(N_{l_0}, (\mu\circ\lambda)_*(\widetilde{Z}^{\mathcal{L}}_{l_0})\right).\end{aligned} \tag{2.7}$$

Recall that $N_{l_0} = (N^s_{l_0}, N^t_{l_0}) \in \mu_*(T_{\lambda(l_0)}\mathcal{M}) \oplus (\mu_*(T_{\lambda(l_0)}\mathcal{M}))^{\perp} = T_{\mu\circ\lambda(l_0)}\mathsf{X}$ and that $N^s_{l_0} = \mu_*(X_{l_0})$, where $X_{l_0} \in T_{\lambda(l_0)}M$ is the unit vector whose equivalence class is $\widetilde{\lambda}(l_0)$. Thus

$$\begin{aligned}g\left(N_{l_0}, (\mu\circ\lambda)_*(\widetilde{Z}^{\mathcal{L}}_{l_0})\right) &= g\left(N^s_{l_0} + N^t_{l_0}, \mu_*(\lambda_*(\widetilde{Z}^{\mathcal{L}}_{l_0}))\right)\\ &= g\left(\mu_*(X_{l_0}), \mu_*(\lambda_*(\widetilde{Z}^{\mathcal{L}}_{l_0}))\right) + g\left(N^t_{l_0}, \mu_*(\lambda_*(\widetilde{Z}^{\mathcal{L}}_{l_0}))\right)\\ &= \overline{g}\left(X_{l_0}, \lambda_*(\widetilde{Z}^{\mathcal{L}}_{l_0})\right) + 0.\end{aligned} \tag{2.8}$$

Since $\widetilde{\lambda}$ is Legendrian, X_{l_0} is $\overline{g}$-orthogonal to $\lambda_*(T_{l_0}\mathcal{L}) \subset T_{\lambda(l_0)}\mathcal{M}$ and hence $\overline{g}\left(X_{l_0}, \lambda_*(\widetilde{Z}^{\mathcal{L}}_{l_0})\right) = 0$. Combining Eqs. (2.6)–(2.8) we have

$$\begin{aligned}g(N_{n_0}, Z_{n_0}) &= \widetilde{g}(\widetilde{N}_{(l_0,t_0)}, \widetilde{Z}_{(l_0,t_0)}) = \widetilde{g}(\widetilde{N}_{(l_0,0)}, \widetilde{Z}_{(l_0,0)})\\ &= g\left(N_{l_0}, (\mu\circ\lambda)_*(\widetilde{Z}^{\mathcal{L}}_{l_0})\right) = \overline{g}\left(X_{l_0}, \lambda_*(\widetilde{Z}^{\mathcal{L}}_{l_0})\right) = 0.\end{aligned} \tag{2.9}$$

This finishes the proof of Statement 1 of the theorem.

Let us prove statement 2. Here the main difficulty is that even when (X, g) is geodesically complete, the geodesic flow on $T\mathsf{X}$ does not seem to give rise to a flow on $ST\mathsf{X}$ or on the subspace of it formed by the null directions, except for some very special (X, g).

Consider the map $\widetilde{\exp}' : T\mathsf{X} \to T\mathsf{X}$ that associates to $v \in T_x\mathsf{X}$ the velocity vector $\gamma'_v(1) \in T_{\gamma_v(1)}\mathsf{X}$ of the unique inextensible geodesic $\gamma_v(t)$ with $\gamma_v(0) = x$ and $\gamma'_v(0) = v$. Put $U' \subset T\mathsf{X}$ to be the (maximal) domain of this map. It is an open set, see [4, Discussion after Lemma 1 in Section 2.8 and Proposition in Section 2.9]. Clearly $\widetilde{\exp}' : U' \to U'$ is a smooth bijection. The inverse map sends $v \in T_x\mathsf{X}$ to $\widetilde{\exp}'(-v)$ and hence is also smooth. Thus $\widetilde{\exp}' : U' \to U'$ is a diffeomorphism.

Put $O \subset U' \subset T\mathsf{X}$ to be (the image of) the zero section of $T\mathsf{X} \to \mathsf{X}$. Put $U = U' \setminus O$. Clearly the restriction $\widetilde{\exp}'|_U$ is a diffeomorphism $U \to U$ that we denote by $\widetilde{\exp}$.

Consider the map $\widetilde{\kappa} : \mathcal{L} \to ST\mathsf{X}$ that is described by the pair: the map $\kappa = \mu\circ\lambda : \mathcal{L} \to \mathsf{X}$ and the vector field $N_l \in T_{\kappa(l)}\mathsf{X}, l \in \mathcal{L}$, along κ. Let us show that $\widetilde{\kappa}$ is an immersion. Take $l \in \mathcal{L}$ and its neighborhood $\mathcal{O} \subset \mathcal{L}$ such that $\lambda(\mathcal{O})$ is contained in an open neighborhood $\mathcal{P} \subset \mathcal{M}$ for which the restriction $\mu|_{\mathcal{P}} : \mathcal{P} \to \mathsf{X}$ is an embedding. It suffices to show that $\widetilde{\kappa}|_{\mathcal{O}}$ is an immersion. The restriction of the bundle $\mathrm{pr}_{\mathcal{M}} : ST\mathcal{M} \to \mathcal{M}$ to $\mathcal{P} \subset \mathcal{M}$ gives the S^{m-1}-bundle $ST\mathcal{P} \to \mathcal{P}$. The restriction of $\mathrm{pr}_{\mathsf{X}} : ST\mathsf{X} \to \mathsf{X}$ to $\mu(\mathcal{P})$ gives the S^m-bundle $ST\mathsf{X}|_{\mu(\mathcal{P})} \to \mu(\mathcal{P})$. The embedding $\mu|_{\mathcal{P}}$ induces the natural bundle map

$$\begin{array}{ccc} ST\mathcal{P} & \xrightarrow{\ i\ } & ST\mathsf{X}|_{\mu(\mathcal{P})} \\ \downarrow & & \downarrow \\ \mathcal{P} & \xrightarrow{\mu|_{\mathcal{P}}} & \mu(\mathcal{P}). \end{array} \tag{2.10}$$

For $p \in \mathcal{P}$ put $V_p \in T_{\mu(p)}\mathsf{X}$ to be the unique future pointing timelike vector such that $g(V_p, V_p) = -1$ and V_p is g-orthogonal to $\mu_*(T_p\mathcal{M})$. Put $V, -V \subset ST\mathsf{X}|_{\mu(\mathcal{P})}$ to be the images of the two sections of $ST\mathsf{X}|_{\mu(\mathcal{P})} \to \mu(\mathcal{P})$ that send $\mu(p)$ to the direction of V_p and to the direction of $-V_p$, respectively. The direct sum decomposition $T_{\mu(p)}\mathsf{X} = \mu_*(T_p\mathcal{P}) \oplus \mathrm{span}(V_p)$ induces the natural fiber preserving smooth map $\pi : ST\mathsf{X}|_{\mu(\mathcal{P})} \setminus (V \sqcup -V) \to ST\mathcal{P}$. For all $l \in \mathcal{O}$ we have $\mu_*(X_l) + V_{\mu(l)} = N_l$. Thus the maps $i \circ \widetilde{\lambda}|_{\mathcal{O}}$ and $\pi \circ \widetilde{\kappa}|_{\mathcal{O}} : \mathcal{O} \to ST\mathcal{P}$ are equal. Since i is an embedding, $\widetilde{\lambda}$ is an immersion, and π is smooth, we get that $\widetilde{\kappa}|_{\mathcal{O}}$ is an immersion, and hence $\widetilde{\kappa}$ is an immersion. Put $\widehat{\kappa} : \mathcal{L} \to T\mathsf{X} \setminus O$ to be the map described by the pair κ and the vector field N_l, $l \in \mathcal{L}$, along κ. Since $\widetilde{\kappa}$ is immersion and it is a composition of $\widehat{\kappa}$ and the smooth quotient map $T\mathsf{X} \setminus O \to ST\mathsf{X}$, we get that $\widehat{\kappa}$ is an immersion.

Define the map $\widehat{\nu} : \mathcal{N} \to T\mathsf{X}$ by sending $(l, t) \in \mathcal{N} \subset \mathcal{L}\times\mathbb{R}$ to $\gamma_l'(t) \in T_{\nu(l,t)}\mathsf{X}$. Let us show that $\widehat{\nu}$ is an immersion. Put $\mathcal{N}^+, \mathcal{N}^-, \mathcal{N}^0 \subset \mathcal{N} \subset \mathcal{L} \times \mathbb{R}$ to be the subsets formed by points (l, t) whose t coordinate is respectively greater than zero, less than zero, and is equal to zero.

Clearly $\mathcal{N}^+$ is open and $\widehat{\nu}(l, t) = \widetilde{\exp}(tN_l)$, for all $(l, t) \in \mathcal{N}^+$. Since $\widehat{\kappa} : \mathcal{L} \to T\mathsf{X} \setminus O$ is an immersion, we get that the map $\beta^+ : \mathcal{N}^+ \to T\mathsf{X} \setminus O$ that sends $(l, t) \in \mathcal{N}^+$ to tN_l is an immersion. Since $\widetilde{\exp}$ is a diffeomorphism, we get that $\widehat{\nu}$ is an immersion at all points of $\mathcal{N}^+$. Similarly one gets that $\widehat{\nu}$ is an immersion at all points of $\mathcal{N}^-$.

Take $(l, 0) \in N^0$ and a non-zero tangent vector $(v_{\mathcal{L}}, v_{\mathbb{R}}) \in T_{(l,0)}\mathcal{N} \subset T_{(l,0)}(\mathcal{L} \times \mathbb{R}) = T_l\mathcal{L} \oplus T_0\mathbb{R} = T_l\mathcal{L} \oplus \mathbb{R}$. Then $\widehat{\nu}_*(v_{\mathcal{L}}, v_{\mathbb{R}}) \in T_{N_l}(T\mathsf{X})$ and $(\tau_{T\mathsf{X}})_* \circ \widehat{\nu}_*(v_{\mathcal{L}}, v_{\mathbb{R}}) = (\tau_{T\mathsf{X}})_* \circ \widehat{\kappa}_*(v_{\mathcal{L}}) + v_{\mathbb{R}}N_l \in T_{\nu(l,0)}\mathsf{X}$. Since $(\tau_{T\mathsf{X}})_* \circ \widehat{\kappa}_*(v_{\mathcal{L}}) \in \mu_*(T_{\lambda(l)}\mathcal{M})$ and $N_l \notin \mu_*(T_{\lambda(l)}\mathcal{M})$, we get that $\widehat{\nu}_*(v_{\mathcal{L}}, v_{\mathbb{R}}) \neq 0$ if $v_{\mathbb{R}} \neq 0$. On the other hand $\widehat{\nu}_*(v_{\mathcal{L}}, 0) = \widehat{\kappa}(v_{\mathcal{L}})$ is non-zero since $\widehat{\kappa}$ is an immersion. Thus $\widehat{\nu}$ is an immersion at all the points of $\mathcal{N}^0$.

Let $q : T\mathsf{X} \setminus O \to ST\mathsf{X}$ be the quotient map by the action of $\mathbb{R}^+$ that we used to define $ST\mathsf{X}$. Clearly $\widetilde{\nu} = q \circ \widehat{\nu}$. Since $\widehat{\nu}$ is an immersion, to prove that $\widetilde{\nu}$ is an immersion it suffices to show that for every $\overline{n} \in \mathcal{N}$ and non-zero $v \in T_{\overline{n}}\mathcal{N}$ the non-zero vector $(\widehat{\nu})_*(v)$ is not tangent to the $\mathbb{R}^+$-fiber of q containing $\widehat{\nu}(\overline{n})$.

We prove this by considering three cases: $\overline{n} \in \mathcal{N}^+$, $\overline{n} \in \mathcal{N}^-$, and $\overline{n} \in \mathcal{N}^0$.

Assume that $\overline{n} = (\bar{l}, \bar{t}) \in \mathcal{N}^+$ and that $\widehat{\nu}_*(v)$ is tangent to the $\mathbb{R}^+$-fiber of q containing $\widehat{\nu}(\overline{n})$. Let $\alpha : (-\epsilon, \epsilon) \to T\mathsf{X}\setminus O$ defined by $\alpha(\tau) = \widehat{\nu}(\overline{n}) + \tau\widehat{\nu}(\overline{n})$ be the parameterization of a small part of the $\mathbb{R}^+$-fiber of q that contains $\widehat{\nu}(\overline{n})$. Since $\widetilde{\exp}$ is a diffeomorphism, we get that $(\widetilde{\exp})_*^{-1} \circ \widehat{\nu}_*(v)$ is a non-zero vector tangent to the curve $\widetilde{\alpha} = \widetilde{\exp}^{-1} \circ \alpha$ at $\widetilde{\exp}^{-1} \circ \alpha(0)$.

Let $\gamma(t)$ be the null geodesic such that $\gamma(0) = \mu \circ \lambda(\bar{l})$ and $\gamma'(0) = \bar{t}N_{\bar{l}}$. Since $\widehat{\nu}|_{\mathcal{N}^+} = \widetilde{\exp} \circ \beta^+$ we get that $\gamma(1) = \tau_{\mathsf{X}}(\widehat{\nu}(\overline{n}))$ and $\gamma'(1) = \widehat{\nu}(\overline{n})$. From the definition of $\widetilde{\exp}$ we get that $\widetilde{\alpha}(\tau) = \widetilde{\exp}^{-1}(\alpha(\tau)) = (\tau + 1)\gamma'(-\tau) \in T_{\gamma(-\tau)}\mathsf{X}$, for all $\tau \in (-\epsilon, \epsilon)$.

Now $\widetilde{\exp}_*^{-1} \circ \widehat{\nu}_*(v)$ is a non-zero vector tangent to the immersed submanifold $S = \{tN_l | t \in \mathbb{R}, l \in \mathcal{L}\} \subset T\mathsf{X}$ at the point $\bar{t}N_{\bar{l}}$. Since $\tau_{\mathsf{X}}(tN_l) = \mu \circ \lambda(l)$ for all $t \in \mathbb{R}, l \in \mathcal{L}$, we get that $(\tau_{\mathsf{X}})_* \left(\widetilde{\exp}_*^{-1} \circ \widehat{\nu}_*(v)\right) \in \mu_*(T_{\lambda(\bar{l})}\mathcal{M})$. Clearly $(\tau_{\mathsf{X}})_*(\widetilde{\alpha}'(0)) = -\gamma'(0) = -\bar{t}N_{\bar{l}} \notin \mu_*(T_{\lambda(\bar{l})}\mathcal{M})$. Thus $\widetilde{\exp}_*^{-1} \circ \widehat{\nu}_*(v)$ is not tangent to $\widetilde{\alpha}$ at $\widetilde{\alpha}(0)$ and $\widetilde{\nu}$ is an immersion at $\overline{n} \in \mathcal{N}^+$.

Hence $\widetilde{\nu}$ is an immersion at all the points of $\mathcal{N}^+$. Similarly one gets that $\widetilde{\nu}$ is an immersion at all the points of $\mathcal{N}^-$.

Let $\overline{n} = (\bar{l}, 0)$ be a point of $\mathcal{N}^0$ and let $(v_{\mathcal{L}}, v_{\mathbb{R}}) \in T_{(\bar{l},0)}\mathcal{N} = T_{\bar{l}}\mathcal{L} \oplus T_0\mathbb{R} = T_{\bar{l}}\mathcal{L} \oplus \mathbb{R}$ be a non-zero tangent vector. Let us show that $\widehat{\nu}_*(v_{\mathcal{L}}, v_{\mathbb{R}})$ is not tangent to the $\mathbb{R}^+$-fiber of q containing $\widehat{\nu}(\bar{l}, 0) = N_{\bar{l}}$. Note that $(\tau_{T\mathsf{X}})_*$ applied to any vector tangent to the $\mathbb{R}^+$-fiber of q is zero, while, as we discussed above, $(\tau_{T\mathsf{X}})_* \circ \widehat{\nu}_*(v_{\mathcal{L}}, v_{\mathbb{R}}) = (\tau_{T\mathsf{X}})_* \circ \widehat{\kappa}_*(v_{\mathcal{L}}) + v_{\mathbb{R}}N_{\bar{l}}$ is non-zero for $v_{\mathbb{R}} \neq 0$. This give the proof for vectors $(v_{\mathcal{L}}, v_{\mathbb{R}})$ with non-zero $v_{\mathbb{R}}$.

Note that $\widehat{\nu}_*(v_{\mathcal{L}}, 0) = \widehat{\kappa}_*(v_{\mathcal{L}})$. Clearly $\widetilde{\kappa} = q \circ \widehat{\kappa}$ and since $\widetilde{\kappa}$ is an immersion we get that $q_* \circ \widehat{\kappa}_*(v_{\mathcal{L}}) = q_* \circ \widehat{\nu}_*(v_{\mathcal{L}}, 0)$ is non-zero for every non-zero $v_{\mathcal{L}}$. On the other hand, q_* applied to any vector tangent to the $\mathbb{R}^+$-fiber of q is zero. Thus $\widetilde{\nu}$ is an immersion at all the points of $\mathcal{N}^0 \subset \mathcal{N}$ and hence $\widetilde{\nu} : \mathcal{N} \to ST\mathsf{X}$ is an immersion. □

Remark 2.2. Let $\nu : \mathcal{N}^m \to \mathsf{X}^{m+1}$ be a mapped null hypersurface and let $h : \mathcal{N}' \to \mathcal{N}$ be a diffeomorphism. Then clearly $\nu \circ h : \mathcal{N}' \to \mathsf{X}$ is a mapped null hypersurface.

If the natural map $\widetilde{\nu} : \mathcal{N}^m \to ST\mathsf{X} = ST^*\mathsf{X}$ is an immersion, then it is a Legendrian immersion and the map $\widetilde{\nu \circ h} : \mathcal{N}' \to ST\mathsf{X} = ST^*\mathsf{X}$ associated with the mapped null hypersurface $\nu \circ h : \mathcal{N}' \to \mathsf{X}$ also is a Legendrian immersion.

Similarly if $U \subset \mathcal{N}$ is open, then $\nu|_U : U \to \mathsf{X}$ is a mapped null hypersurface. Note that if $\nu|_U$ is an embedding, then $\nu|_U : U \to \mathsf{X}$ is an embedded null hypersurface.

3. From mapped null hypersurfaces to Legendrian maps

Let (X^{m+1}, g) be a space–time. Let $\nu : \mathcal{N}^m \to \mathsf{X}^{m+1}$ be a mapped null hypersurface, let $\mu : \mathcal{M}^m \to \mathsf{X}^{m+1}$ be an immersed spacelike hypersurface, and let $\mathcal{L}_{\mu,\nu}$ be the pull back of the maps μ and ν. We will show that μ and ν canonically define a Legendrian map $\widetilde{\lambda}_{\mu,\nu} : \mathcal{L}_{\mu,\nu} \to ST^*\mathcal{M}$ of the $(m-1)$-dimensional pull-back manifold and that $\mathrm{Im}(\mathrm{pr}_{\mathcal{M}} \circ \widetilde{\lambda}_{\mu,\nu}) = \mathrm{Im}\,\mu^{-1}(\mathrm{Im}\,\mu \cap \mathrm{Im}\,\nu)$.

We will also show that if the map $\widetilde{\nu} : \mathcal{N}^m \to ST\mathsf{X}$ associated to ν is an immersion, then $\widetilde{\lambda}_{\mu,\nu}$ is a Legendrian immersion. In this case the singularities of $\mathrm{pr}_{\mathcal{M}} \circ \widetilde{\lambda}_{\mu,\nu}$ are Legendrian singularities. In particular, this is so when ν is the mapped null hypersurface arising from a Legendrian immersion $\widetilde{\lambda}' : \mathcal{L}' \to ST^*\mathcal{M}'$ for some immersed spacelike hypersurface $\mu' : \mathcal{M}' \to \mathsf{X}$, see Theorem 2.1.

The Lorentz metric g allows us to identify $ST\mathsf{X}$ with $ST^*\mathsf{X}$. Let $N_n \in T_{\nu(n)}\mathsf{X}$, $n \in \mathcal{N}$, be a smooth nowhere zero null vector field along ν such that for all $n \in \mathcal{N}$ the equivalence class of N_n is $\widetilde{\nu}(n) \in ST\mathsf{X} = ST^*\mathsf{X}$. For $n \in \mathcal{N}$, put $\theta_n \in T^*_{\nu(n)}\mathsf{X}$ to be the non-zero covector such that $\theta_n(v) = g(N_n, v)$, for all $v \in T_{\nu(n)}\mathsf{X}$. We get the smooth nowhere zero covector field θ_n, $n \in \mathcal{N}$, along ν such that for all $n \in \mathcal{N}$ the equivalence class of θ_n is $\widetilde{\nu}(n) \in ST^*\mathsf{X} = ST\mathsf{X}$.

Consider the pull-back diagram

$$\begin{array}{ccc} \mathcal{L}_{\mu,\nu} & \xrightarrow{\lambda_{\mu,\nu}} & \mathcal{M}^m \\ \big\downarrow j & & \big\downarrow \mu \\ \mathcal{N}^m & \xrightarrow{\nu} & \mathsf{X}^{m+1}. \end{array} \tag{3.1}$$

By definition of the pull back $\mathcal{L}_{\mu,\nu} = \{(m,n) \in \mathcal{M} \times \mathcal{N} | \mu(m) = \nu(n)\} \subset \mathcal{M} \times \mathcal{N}$. Choose $(m,n) \in \mathcal{L}_{\mu,\nu}$. Since μ is an immersion, $\mu_*(T_m\mathcal{M})$ is m-dimensional. Since ν is a mapped null hypersurface and by definition of $\widetilde{\nu}$, the non-zero vector $N_n \in \nu_*(T_n\mathcal{N})$ is null. Since μ is spacelike, all the non-zero vectors in $\mu_*(T_m\mathcal{M})$ are spacelike, and hence $N_n \notin \mu_*(T_m\mathcal{M})$. For dimension reasons we get that the minimal linear subspace of $T_{\nu(n)}\mathsf{X} = T_{\mu(m)}\mathsf{X}$ that contains $\mu_*(T_m\mathcal{M}) \cup \nu_*(T_n\mathcal{N})$ is equal to $T_{\nu(n)}\mathsf{X} = T_{\mu(m)}\mathsf{X}$. Thus μ and ν are transverse and hence $\mathcal{L}_{\mu,\nu}$ is an $(m-1)$-dimensional smooth embedded submanifold of $\mathcal{M} \times \mathcal{N}$.

Clearly $\lambda_{\mu,\nu}(m,n) = m$ and $j(m,n) = n$, for $(m,n) \in \mathcal{L}_{\mu,\nu}$. We define the smooth covector field $\phi_l \in T^*_{\lambda(l)}\mathcal{M}$, $l \in \mathcal{L}_{\mu,\nu}$, along $\lambda_{\mu,\nu}$ as follows. For $l = (m,n) \in \mathcal{L}_{\mu,\nu}$ and $v \in T_{\lambda(l)}\mathcal{M} = T_m\mathcal{M}$ put $\phi(v) = \theta_n(\mu_*(v))$. Recall that the covector θ_n is non-zero, $\theta_n|_{\nu_*(T_n\mathcal{N})}$ is zero, and $T_{\mu(m)}\mathsf{X} = T_{\nu(n)}\mathsf{X}$ is the linear span of $\mu_*(T_m\mathcal{M}) \cup \nu_*(T_n\mathcal{N})$. Thus the covector field ϕ_l, $l \in \mathcal{L}_{\mu,\nu}$, along $\lambda_{\mu,\nu}$ is nowhere zero. Hence the pair: $\lambda_{\mu,\nu}$ and the covector field ϕ_l, $l \in \mathcal{L}_{\mu,\nu}$, along $\lambda_{\mu,\nu}$ define a map $\widetilde{\lambda}_{\mu,\nu} : \mathcal{L}_{\mu,\nu} \to ST^*\mathcal{M} = ST\mathcal{M}$. It is easy to see that the map $\widetilde{\lambda}_{\mu,\nu}$ does not depend on the choice of the vector field N_n, $n \in \mathcal{N}$, along ν from which we started the construction.

Theorem 3.1. *Let (X^{m+1}, g) be a space–time, let $\nu : \mathcal{N}^m \to \mathsf{X}^{m+1}$ be a mapped null hypersurface, and let $\mu : \mathcal{M}^m \to \mathsf{X}^{m+1}$ be an immersed spacelike hypersurface. Let $\mathcal{L}_{\mu,\nu}$ be the smooth $(m-1)$-dimensional manifold that is the pull back of μ and ν. Let $\widetilde{\lambda}_{\mu,\nu} : \mathcal{L}_{\mu,\nu} \to ST^*\mathcal{M}$ be the map constructed above. Then*

1. *The map $\widetilde{\lambda}_{\mu,\nu} : \mathcal{L}_{\mu,\nu} \to ST^*\mathcal{M}$ is a Legendrian map and* $\mathrm{Im}(\mathrm{pr}_{\mathcal{M}} \circ \widetilde{\lambda}_{\mu,\nu}) = \mathrm{Im}\,\mu^{-1}(\mathrm{Im}\,\mu \cap \mathrm{Im}\,\nu)$.
2. *If the map $\widetilde{\nu} : \mathcal{N} \to ST\mathsf{X}$ that is naturally associated with ν is an immersion, then the map $\widetilde{\lambda}_{\mu,\nu} : \mathcal{L}_{\mu,\nu} \to ST^*\mathcal{M}$ is a Legendrian immersion.*

Proof. *Let us prove statement* 1 *of the theorem.* The fact that $\mathrm{Im}(\mathrm{pr}_{\mathcal{M}} \circ \widetilde{\lambda}_{\mu,\nu}) = \mathrm{Im}\,\mu^{-1}(\mathrm{Im}\,\mu \cap \mathrm{Im}\,\nu)$ is clear from the construction of $\widetilde{\lambda}_{\mu,\nu}$.

To see that $\widetilde{\lambda}_{\mu,\nu}$ is a Legendrian map it suffices to show that $\phi_l\left((\lambda_{\mu,\nu})_*(v)\right) = 0$ for every $l \in \mathcal{L}_{\mu,\nu}$ and $v \in T_l\mathcal{L}_{\mu,\nu}$.

By definition of ϕ_l we have $\phi_l\left((\lambda_{\mu,\nu})_*(v)\right) = \theta_n\left((\mu \circ \lambda_{\mu,\nu})_*(v)\right)$. Since the diagram (3.1) is commutative, we have $\theta_n\left((\mu \circ \lambda_{\mu,\nu})_*(v)\right) = \theta_n\left(\nu_*(j_*(v))\right)$. By definition of $\widetilde{\nu}$ we have that $\theta_n|_{\mathrm{Im}\,\nu_*(T_n\mathcal{N})} = 0$. Thus $\phi_l\left((\lambda_{\mu,\nu})_*(v)\right) = \theta_n\left(\nu_*(j_*(v))\right) = 0$.

To prove statement 2 *of the theorem* we will show that for every $\bar{l} = (\bar{m}, \bar{n}) \in \mathcal{L}_{\mu,\nu}$ the map $\widetilde{\lambda}_{\mu,\nu}$ is an immersion at $\bar{l}$. Put $\mathcal{P} \subset \mathcal{M}$ to be an open neighborhood such that $\bar{m} \in \mathcal{P}$ and $\mu|_{\mathcal{P}}$ is an embedding.

Put $\mathcal{L} = \mathcal{L}_{\mathcal{P}} = \{(m,n) \in \mathcal{M} \times \mathcal{N} | \mu(m) = \nu(n) \text{ and } m \in \mathcal{P}\} \subset \mathcal{L}_{\mu,\nu}$. We will denote $\widetilde{\lambda}_{\mu,\nu}|_{\mathcal{L}} : \mathcal{L} \to ST^*\mathcal{P} \subset ST\mathcal{M}$ by $\widetilde{\lambda}_{\mathcal{L}}$ and we will denote $\lambda_{\mu,\nu}|_{\mathcal{L}} : \mathcal{L} \to \mathcal{P}$ by $\lambda_{\mathcal{L}}$. It suffices to show that $\widetilde{\lambda}_{\mathcal{L}}$ is an immersion at $\bar{l} \in \mathcal{L}$.

Take a non-zero vector $v = (v_{\mathcal{P}}, v_{\mathcal{N}}) \in T_{\bar{l}}\mathcal{L} \subset T_{(\bar{m},\bar{n})}(\mathcal{P} \times \mathcal{N}) = T_{\bar{m}}\mathcal{P} \oplus T_{\bar{n}}\mathcal{N}$. From the construction of $\lambda_{\mathcal{L}}$ and $\widetilde{\lambda}_{\mathcal{L}}$ we get that $(\mathrm{pr}_{\mathcal{P}})_* \circ (\widetilde{\lambda}_{\mathcal{L}})_*(v) = v_{\mathcal{P}}$. Thus $(\widetilde{\lambda}_{\mathcal{L}})_*(v) \neq \mathbf{0}$ if $v_{\mathcal{P}} \neq \mathbf{0}$. Hence it suffices to show that $(\widetilde{\lambda}_{\mathcal{L}})_*(\mathbf{0}, v_{\mathcal{N}}) \neq \mathbf{0}$ for $(\mathbf{0}, v_{\mathcal{N}}) \in T_{\bar{l}}\mathcal{L}$ with non-zero $v_{\mathcal{N}}$.

The embedding $\mu|_{\mathcal{P}} : \mathcal{P} \to \mathsf{X}$ induces the diffeomorphism $ST\mu|_{\mathcal{P}} : ST\mathcal{P} \to ST\mu(\mathcal{P})$ onto the spherical tangent bundle of $\mu(\mathcal{P})$.

Consider the map $\widetilde{\nu} \circ j : \mathcal{L} \to ST\mathsf{X}$ that maps $(m,n) \in \mathcal{L}$ to the equivalence class of $N_n \in T_{\nu(n)}\mathsf{X}$. Since $\mathcal{L}$ is the pull back of ν and $\mu|_{\mathcal{P}}$ we get that $\nu(n) \in \mu(\mathcal{P})$ for all $(m,n) \in \mathcal{L}$. Thus $\widetilde{\nu} \circ j(\mathcal{L})$ is in the total space $ST\mathsf{X}|_{\mu(\mathcal{P})}$ of the restriction of the S^m-bundle $ST\mathsf{X} \to \mathsf{X}$ to $\mu(\mathcal{P}) \subset \mathsf{X}$. Consider the S^{m-1}-bundle $\mathcal{C} \to \mathsf{X}$ whose total space $\mathcal{C} \subset ST\mathsf{X}$ is formed by the future pointing null directions. Clearly $\widetilde{\nu}(\mathcal{N}) \subset \mathcal{C}$. Thus $\widetilde{\nu} \circ j(\mathcal{L})$ is in the total space $\mathcal{C}|_{\mu(\mathcal{P})}$ of the restriction of the bundle $\mathcal{C} \to \mathsf{X}$ to $\mu(\mathcal{P})$.

Put $T\mathsf{X}|_{\mu(\mathcal{P})}$ to be the total space of the restriction to $\mu(\mathcal{P})$ of the bundle $T\mathsf{X} \to \mathsf{X}$. The g-orthogonal projection $T\mathsf{X}|_{\mu(\mathcal{P})} \to T\mu(\mathcal{P})$ induces the diffeomorphism $\delta : \mathcal{C}|_{\mu(\mathcal{P})} \to ST\mu(\mathcal{P})$.

The Riemannian metric $\overline{g} = \mu^* g$ on $T\mathcal{P}$ allows us to identify $ST\mathcal{P}$ with $ST^*\mathcal{P}$. For $l = (m, n) \in \mathcal{L}$ put $V_l \in T_{\lambda_{\mathcal{L}}(l)}\mathcal{P} = T_m\mathcal{P}$ to be the unique vector such that $\phi_l(w) = \overline{g}(V_l, w)$, for all $w \in T_m\mathcal{P}$. From the construction of ϕ_l it is easy to see that $\mu_*(V_l)$ is the g-orthogonal projection of $N_n \in T_{\nu(n)}\mathsf{X}$ to $\mu_*(T_m\mathcal{P}) \subset T_{\nu(n)}\mathsf{X}$. One verifies that the equivalence class of V_l in $ST\mathcal{P} = ST^*\mathcal{P}$ is $\widetilde{\lambda}_{\mathcal{L}}(l)$.

Thus we have that the maps $\widetilde{\lambda}_{\mathcal{L}} : \mathcal{L} \to ST\mathcal{P} = \mathcal{S}T^*P$ and $(ST\mu|_{\mathcal{P}})^{-1} \circ \delta \circ \widetilde{\nu} \circ j|_{\mathcal{L}} : \mathcal{L} \to ST\mathcal{P}$ are equal. Thus if $v = (\mathbf{0}, v_{\mathcal{N}}) \in T_{\bar{l}}\mathcal{L}$ is a non-zero vector, then we have $(\widetilde{\lambda}_{\mathcal{L}})_*(v) = (ST\mu|_{\mathcal{P}})_*^{-1} \circ \delta_* \circ \widetilde{\nu}_* \circ (j|_{\mathcal{L}})_*(\mathbf{0}, v_{\mathcal{N}}) = (ST\mu|_{\mathcal{P}})_*^{-1} \circ \delta_* \circ \widetilde{\nu}_*(v_{\mathcal{N}})$. Since $v_{\mathcal{N}} \neq \mathbf{0}$, $\widetilde{\nu}$ is an immersion, and $(ST\mu_{\mathcal{P}})^{-1}$, δ are diffeomorphisms, we get that $(\widetilde{\lambda}_{\mathcal{L}})_*(v) \neq \mathbf{0}$. Hence $\widetilde{\lambda}_{\mu,\nu}$ is an immersion. □

Example 3.2 (*Null Cone*). Let (X, g) be a space–time. For $x \in \mathsf{X}$ put C_x^+ (respectively C_x^-) to be the hemicone of future pointing (respectively past pointing) null vectors in $T_x\mathsf{X}$. Put $\mathcal{C}_x^+ \subset C_x^+$ and $\mathcal{C}_x^- \subset C_x^-$ to be the maximal open subsets on which $\exp_x$ is well defined.

Choose a (possibly small) immersed spacelike hypersurface $\mu : \mathcal{M}^m \to \mathsf{X}^{m+1}$ such that $x = \mu(\overline{x})$ for some $\overline{x} \in \mathcal{M}$ and let $\overline{g}$ be the induced Riemannian metric on $\mathcal{M}$. Let $\widetilde{\lambda} : S^{m-1} \to ST^*\mathcal{M}$ be a Legendrian embedding whose image is the S^{m-1}-fiber over the point $\overline{x}$. Let $\nu : \mathcal{N} \to \mathsf{X}$ be the mapped null hypersurface from Theorem 2.1 constructed using the above $\mu, \widetilde{\lambda}$ and $\mathcal{L} = S^{m-1}$. By Theorem 2.1 the natural map $\widetilde{\nu} : \mathcal{N} \to ST\mathsf{X}$ is a Legendrian immersion.

Let $N_l \in T_{\mu\circ\lambda(l)}\mathsf{X} = T_x\mathsf{X}$, $l \in S^{m-1}$, be the future pointing null vector field along $\mu \circ \lambda$ that we used to construct ν. Consider the map $h : S^{m-1} \times \mathbb{R} \to (C_x^+ \sqcup C_x^- \sqcup \mathbf{0}) \subset T_x\mathsf{X}$ defined by $h(l, t) = tN_l$. Put $\mathcal{N}^+$ (respectively $\mathcal{N}^-$) to be the open subset of $\mathcal{N} \subset S^{m-1} \times \mathbb{R}$ consisting of all the points with the positive (respectively negative) $\mathbb{R}$-coordinate. Clearly $h : \mathcal{N}^+ \to \mathcal{C}^+$ and $h : \mathcal{N}^- \to \mathcal{C}^-$ are diffeomorphisms and $\exp_x(h(l, t)) = \nu(l, t)$, for all $(l, t) \in \mathcal{N}^\pm$.

Combining this with Remark 2.2 we get that $\exp_x : \mathcal{C}_x^+ \to \mathsf{X}$ and $\exp_x : \mathcal{C}_x^- \to \mathsf{X}$ are mapped null hypersurfaces, i.e. the exponential of the future and of the past null hemicones at x are mapped null hypersurfaces on the maximal open subsets where they are defined. In particular if an open $U \subset \mathcal{C}^\pm$ is such that $\exp_x|_U$ is an embedding, then $\exp_x : U \to \mathsf{X}$ is an embedded null hypersurface.[1]

Moreover by Theorem 2.1 the natural maps $\mathcal{C}_x^+ \to ST\mathsf{X} = ST^*\mathsf{X}$ and $\mathcal{C}_x^- \to ST\mathsf{X} = ST^*\mathsf{X}$ are Legendrian immersions.

Let $\mu' : \mathcal{M}' \to \mathsf{X}$ be an immersed spacelike hypersurface. Then by Theorem 3.1 the map $\exp_x|_{\mathcal{C}_x^+} : \mathcal{C}_x^+ \to \mathsf{X}$ defines the Legendrian immersion $\widetilde{\lambda}_{\mu',\exp_x|_{\mathcal{C}_x^+}} : \mathcal{L}_{\mu',\exp_x|_{\mathcal{C}_x^+}} \to ST\mathcal{M}'$ such that $\operatorname{Im}(\operatorname{pr}_{\mathcal{M}'} \circ \widetilde{\lambda}_{\mu',\exp_x|_{\mathcal{C}_x^+}}) = \exp_x(\mathcal{C}_x^+) \cap \mu'(\mathcal{M}')$. Thus the intersection of the future null cone of x with the spacelike immersed hypersurface $\mu(\mathcal{M}')$ is naturally parameterized by the projection to $\mathcal{M}'$ of the Legendrian immersion to $ST^*\mathcal{M}'$.

Similarly one get that the intersection of the past null cone of x with the spacelike immersed hypersurface $\mu(\mathcal{M}')$ is also naturally parameterized by the projection to $\mathcal{M}'$ of a Legendrian immersion to $ST^*\mathcal{M}'$.

Acknowledgments

The author is very grateful to Robert Low, Yuli Rudyak, Miguel Sanchez, and David Webb for useful discussions.

Appendix. Low's [9] results on null congruences and Legendrian submanifolds of the space of null geodesics in globally hyperbolic and strongly causal (X^{3+1}, g)

Recall a few more Lorentzian geometry definitions and facts. An open set in (X^{m+1}, g) is *causally convex* if its intersection with every non-spacelike curve is connected and (X, g) is *strongly causal* if every point in it has

[1] In the work of Lerner [7, Lemma 2] it is proved that the exponential of the future null cone of x is an embedded null hypersurface when restricted to the preimage under $\exp_x$ of a simple neighborhood of x. We did not find more general statements about null cones giving rise to embedded null hypersurfaces in the literature. Miguel Sanchez pointed to us that the fact that $\exp_x|_U$ is an embedded null hypersurface also follows from the Gauss Lemma for Lorentzian manifolds [11] and we thank him for this remark.

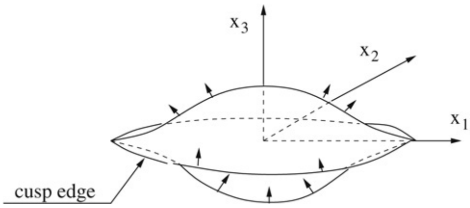

Fig. A.1. Legendrian submanifold with a "flying saucer" projection.

arbitrarily small causally convex neighborhoods. A strongly causal space–time (X, g) is *globally hyperbolic* if for every $x_1, x_2 \in \mathsf{X}$ the set of all $x \in \mathsf{X}$ such that there exists a piecewise smooth non-spacelike curve from x_1 to x_2 through x is compact.

A *Cauchy surface* M is a subset of a space–time X such that for every inextensible non-spacelike curve $\gamma(t)$ in X there exists exactly one $t_0 \in \mathbb{R}$ with $\gamma(t_0) \in M$. A space–time is globally hyperbolic if and only if it admits a Cauchy surface, see [5, pp. 211–212]. Geroch [3] showed that globally hyperbolic (X, g) are rather simple topologically and they are homeomorphic to a product of $\mathbb{R}$ and a Cauchy surface. Bernal and Sanchez [2] showed that every globally hyperbolic space–time (X^{m+1}, g) admits a smooth spacelike Cauchy surface and moreover X is in fact diffeomorphic to a product of $\mathbb{R}$ and this Cauchy surface.

Put $\mathfrak{N} = \mathfrak{N}_{(\mathsf{X},g)}$ to be the space of all future directed null geodesics in (X, g) up to an affine reparameterization. In general $\mathfrak{N}$ is not a manifold. However for globally hyperbolic (X, g), the space $\mathfrak{N}$ is a smooth contact manifold contactomorphic to the spherical cotangent bundle ST^*M of a smooth spacelike Cauchy surface $M^m \subset \mathsf{X}^{m+1}$. This fact was proved by Low [9, Corollary 1, Lemma 2, Corollary 2] for $(3+1)$-dimensional globally hyperbolic (X^{3+1}, g). This result and the techniques, Low used to get it, generalize to globally hyperbolic space–times of all dimensions, see Natario and Tod [10, pp. 252–253].

Since the Cauchy surface M is spacelike we can identify STM and ST^*M. Under the contactomorphism $\mathfrak{N} \to ST^*M$ a null geodesic γ is mapped to the point of $ST^*M = STM$ that is the direction of the g-orthogonal projection to M of the velocity vector of γ at the intersection point of γ with M.

Low [9] observed strong and fascinating relations between null congruences and Legendrian submanifolds of $\mathfrak{N}$ for $(3+1)$-dimensional globally hyperbolic (X^{3+1}, g). The combination of his [9, Lemma 2, Corollary 3] says that the null congruences orthogonal to a two-dimensional spacelike surface are exactly the Legendrian submanifolds of $ST^*M = \mathfrak{N}$. Unfortunately, if taken literally this statement is false for rather technical reasons.

For example, in order for the Legendrian submanifold to be embedded, rather than immersed, one has to require that no two points of the two-dimensional spacelike surface Σ belong to the same null geodesic that is g-orthogonal to Σ. This would follow automatically if the two-dimensional spacelike surface Σ is a subset of some Cauchy surface. However it is easy to construct examples of two-dimensional embedded spacelike surfaces Σ in globally hyperbolic (X^{3+1}, g) such that there are two points in Σ that belong to the same null geodesic that is g-orthogonal to Σ.

It also is possible to find Legendrian submanifolds of $\mathfrak{N}$ that are not realizable as null congruences orthogonal to a spacelike 2-surface. Consider a globally hyperbolic $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, t) and the Lorentz metric $g = dx_1^2 + dx_2^2 + dx_3^2 - dt^2$. For $\tau \in \mathbb{R}$ define the spacelike Cauchy surface $\mathbb{R}^3_\tau \subset \mathbb{R}^4$ to be the set of all the points whose t-coordinate equals τ. Take a Legendrian submanifold $L \subset ST^*\mathbb{R}^3_0 = ST\mathbb{R}^3_0$ that is described by the projection of L to $\mathbb{R}^3_0$ which is the rotationally symmetric "flying saucer" and the unit length vector field along the projection of L orthogonal to the "saucer", see Fig. A.1. We assume that the cusp edge of the "saucer" is the circle $\{(x_1, x_2, 0, 0) | x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^3_0 \subset \mathbb{R}^4$.

Since $\mathbb{R}^3_0$ is a Cauchy surface, $ST^*\mathbb{R}^3_0$ is identified with $\mathfrak{N}$ and L gives a Legendrian submanifold $L' \subset \mathfrak{N}$. For every τ, the intersection of the corresponding null congruence with $\mathbb{R}^3_\tau$ will have a cusp edge along the circle $\{(x_1, x_2, \tau, \tau) | x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^3_\tau \subset \mathbb{R}^4$ (and possibly other singularities). It is easy to see that the subsets of L' that give rise to the cusp edges $\{(x_1, x_2, 0, 0) | x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^3_0$ and $\{(x_1, x_2, \tau, \tau) | x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^3_\tau$ are equal. One verifies that no open in L' neighborhood of a point in this subset can be realized as a null congruence orthogonal to some embedded (or immersed) two-dimensional spacelike surface in $(\mathbb{R}^4, g)$.

Low [8,9] also remarked that null congruences orthogonal to two-dimensional spacelike surfaces are related to Legendrian submanifolds of $\mathfrak{N}$ for strongly causal (X^{3+1}, g). In this case $\mathfrak{N}$ is a smooth contact manifold that is possibly not Hausdorff.

The contact structure on the space of null geodesics and the symplectic structure on the spaces of timelike and spacelike geodesics in general pseudo-Riemannian manifolds was very recently studied by Khesin and Tabachnikov [6]. In order for their results to apply the pseudo-Riemannian manifold should be such that these spaces of geodesics are manifolds. This imposes very strong restrictions on the pseudo-Riemannian manifolds under consideration.

Our work is motivated by Low's work and it establishes relations between Legendrian and null maps for an arbitrary space–time (X^{m+1}, g), including those space–times for which the space of null geodesics is not a manifold. In particular, Theorem 2.1 shows that for an immersed spacelike hypersurface $\mu : \mathcal{M}^m \to \mathsf{X}^{m+1}$, the null congruence associated to a Legendrian map $\widetilde{\lambda} : \mathcal{L}^{m-1} \to ST^*\mathcal{M}$ gives a mapped null hypersurface $\nu : \mathcal{N}^m \to \mathsf{X}^{m+1}$, and that moreover the natural map $\widetilde{\nu} : \mathcal{N}^m \to ST^*\mathsf{X}$ is a Legendrian immersion if $\widetilde{\lambda}$ is a Legendrian immersion.

Theorem 3.1 says that the intersection of a mapped null hypersurface $\nu : \mathcal{N}^m \to \mathsf{X}^{m+1}$ with any immersed spacelike hypersurface $\mu' : \mathcal{M}' \to \mathsf{X}$ gives a Legendrian map to $ST^*\mathcal{M}'$, and that moreover this map is a Legendrian immersion if $\widetilde{\nu}$ is an immersion. In this case the intersection $(\mu')^{-1}\left(\nu(\mathcal{N}) \cap \mu'(\mathcal{M}')\right)$ is naturally parameterized by a Legendrian immersion to $ST^*\mathcal{M}'$.

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